

Linear Algebra

Exam

Common part

Fall 2024

Questions

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- 1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- 1 points if your answer is incorrect.

Notation (all standard)

- \mathbb{R} denotes the set of real numbers.
- For a matrix A , $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j .
- For a vector $x \in \mathbb{R}^n$, x_i denotes the i th coordinate of x .
- I_m denotes the $m \times m$ identity matrix.
- \mathbb{P}_n is the vector space of polynomials of degree less than or equal to n .
- $\mathbb{R}^{m \times n}$ is the vector space of $m \times n$ matrices.
- The scalar or inner product of vectors $x, y \in \mathbb{R}^n$ is defined as $x \cdot y = x^T y$.
- The length of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x \cdot x}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1: The line that best approximates (in the sense of least squares) the points $(0, 2)$, $(1, 1)$, $(2, 1)$, $(5, 0)$ is

$y = \frac{11}{5} + \frac{3}{5}x$ $y = 2 - \frac{3}{8}x$ $y = \frac{9}{2} - \frac{5}{2}x$ $y = \frac{12}{7} - \frac{5}{14}x$

Question 2: Let $T: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{P}_2$ be the linear transformation defined by

$$T \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{12} - (b_{11} + b_{22})t + b_{21}t^2.$$

Let

$$\underline{v} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right\} \quad \text{and} \quad \underline{w} = \{1, 1 + t^2, t + t^2\}$$

be ordered bases of $\mathbb{R}^{2 \times 2}$ and \mathbb{P}_2 respectively. Let A be the matrix that represents T with respect to the bases \underline{v} of $\mathbb{R}^{2 \times 2}$ and \underline{w} of \mathbb{P}_2 , so that $[T(B)]_{\underline{w}} = A[B]_{\underline{v}}$ for all $B \in \mathbb{R}^{2 \times 2}$. Then A satisfies

$a_{12} = -6$ $a_{12} = 3$ $a_{12} = 2$ $a_{12} = -4$

Question 3: Let $\underline{b} = \{b_1, b_2, b_3\}$ be an ordered basis of \mathbb{R}^3 and let $\underline{c} = \{c_1, c_2, c_3\}$ be an ordered basis of \mathbb{R}^3 built by the vectors

$$c_1 = b_1 - b_2, \quad c_2 = b_1 - b_3 \quad \text{and} \quad c_3 = b_1 + b_2 + b_3.$$

Let P be the change of basis matrix $[v]_{\underline{c}} = P[v]_{\underline{b}}$ for all $v \in \mathbb{R}^3$. Then

$p_{23} = 1$ $p_{23} = 2$ $p_{23} = -\frac{2}{3}$ $p_{23} = 0$

Question 4: Let

$$A = \begin{pmatrix} \pi & 1 & 3 & 1 \\ 2\pi & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

If $B = A^{-1}$ is the inverse of the matrix A , then

$b_{24} = 3$ $b_{21} = 7$ $b_{23} = 5$ $b_{22} = 1$

Question 5: Let a be a real number. The linear system

$$\begin{cases} ax_1 + x_3 = 1 \\ ax_2 + x_4 = 0 \\ x_1 + ax_3 = 1 \\ x_2 + ax_4 = 1 \\ x_1 + x_3 = a \end{cases}$$

has exactly one solution if and only if

$a \in \{-2, 1\}$

$a = -2$

$a \in \{-1, 1\}$

$a \in \{-1, 1, -2\}$

Question 6: Let A be a matrix of size 2×2 so that $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are eigenvectors for the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Then the inverse matrix A^{-1} is

$A^{-1} = \begin{pmatrix} 5/2 & -1 \\ 3 & -1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -2 & -2 \\ -18 & -11 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -2 & 2 \\ 6 & 5 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 5/2 & 1 \\ -3 & -1 \end{pmatrix}$

Question 7: Let $v = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the linear transformation defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y - 5z \\ -2x + y + 7z \\ x - 2z \end{pmatrix}.$$

Then

the vector v is neither in $\text{Ker}(T)$, nor in $\text{Im}(T)$

the vector v is in $\text{Ker}(T)$, but not in $\text{Im}(T)$

the vector v is in $\text{Im}(T)$ and in $\text{Ker}(T)$

the vector v is in $\text{Im}(T)$, but not in $\text{Ker}(T)$

Question 8: Let

$$w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_3 = \begin{pmatrix} -2 \\ 0 \\ 4 \\ 2 \\ -4 \end{pmatrix}$$

be three vectors in \mathbb{R}^5 and let $W = \text{span}\{w_1, w_2, w_3\}$. Applying the Gram-Schmidt algorithm to $\{w_1, w_2, w_3\}$ yields an orthogonal basis $\{v_1, v_2, v_3\}$ of W such that

$v_3 = \begin{pmatrix} -2 \\ 0 \\ 4 \\ 2 \\ -4 \end{pmatrix}$

$v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}$

$v_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 1 \\ -1 \end{pmatrix}$

$v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \\ -1 \end{pmatrix}$

Question 9: The eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

are

$-2, -1$ and 2

-1 and 2

$-1, 1$ and 2

$-2, -1$ and 1

Question 10: Let W be the vector space of symmetric matrices of size 2×2 . The second coordinate of the element $\begin{pmatrix} -3 & 2 \\ 2 & 1 \end{pmatrix} \in W$ with respect to the ordered basis $\left\{ \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\}$ of W is

$-\frac{3}{4}$

$\frac{1}{4}$

-2

1

Question 11: The system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 3 \\ x_2 - 2x_3 + 2x_4 = 2 \\ 4x_1 + 3x_2 + 4x_3 - 3x_4 = 1 \\ 5x_1 + 3x_2 + 3x_3 - x_4 = 0 \end{cases}$$

has a unique solution which satisfies

$x_4 = 1$

$x_4 = \frac{1}{2}$

$x_4 = -2$

$x_4 = -\frac{5}{2}$

Question 12: Let $\alpha \in \mathbb{R}$ and $B \in \mathbb{R}^{4 \times 4}$ defined by

$$B = \begin{pmatrix} 1 & 1 & \alpha & 0 \\ 1 & 1 & 0 & \alpha \\ \alpha & 0 & 1 & 1 \\ 0 & \alpha & 1 & 1 \end{pmatrix}.$$

Then

$\det(B) = \alpha^2(\alpha - 1)^2$

$\det(B) = \alpha(\alpha + 2)(\alpha - 1)^2$

$\det(B) = \alpha^2(\alpha^2 - 4)$

$\det(B) = \alpha(\alpha + 1)(\alpha^2 - 4)$

Question 13: Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}.$$

Then the least squares solution $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ of $Ax = b$ satisfies

$x_1^* = -2$

$x_1^* = 2$

$x_1^* = -1$

$x_1^* = 1$

Question 14: Let W be the subspace of \mathbb{R}^4 defined by

$$W = \{x \in \mathbb{R}^4 : x_1 - x_2 + x_3 - x_4 = 0\}.$$

Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ be a basis of W . Then the orthogonal projection of $v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ onto W is

$\begin{pmatrix} 3/2 \\ 4 \\ 6 \\ 7/2 \end{pmatrix}$

$\begin{pmatrix} 1/2 \\ 2 \\ 3 \\ -1/2 \end{pmatrix}$

$\begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

$\begin{pmatrix} 3/2 \\ 3/2 \\ 7/2 \\ 7/2 \end{pmatrix}$

Second part: true/false questions

For each question, mark the box TRUE if the statement is **always true** or the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 15: The set

$$\{p \in \mathbb{P}_n : p(1)p(2) = 0\}$$

is a subspace of \mathbb{P}_n .

TRUE FALSE

Question 16: Let W be a subspace of \mathbb{R}^n . If $v_1, v_2 \in \mathbb{R}^n$ are vectors whose orthogonal projections onto the subspace W are equal, then the vector $v_1 - v_2$ is orthogonal to W .

TRUE FALSE

Question 17: Let A and B be two matrices of size 3×3 . If $\det(A) = 2$, then

$$\det(BA^{-1}B) = \frac{1}{2} \det(B)^2.$$

TRUE FALSE

Question 18: Let V and W be two vector spaces and let $T, S: V \rightarrow W$ be two linear transformations. If $\text{Ker}(T) = \text{Ker}(S)$, then $\text{Im}(T) = \text{Im}(S)$.

TRUE FALSE

Question 19: If $A \in \mathbb{R}^{n \times n}$ is so that $\det(A) = 0$, then $\lambda = 0$ is an eigenvalue of A .

TRUE FALSE

Question 20: If $A \in \mathbb{R}^{m \times n}$ satisfies $\text{Col}(A) = \mathbb{R}^m$, then for every $b \in \mathbb{R}^m$ the linear system $Ax = b$ has a unique solution.

TRUE FALSE

Question 21: Let V be a vector space and let $T: V \rightarrow \mathbb{R}$ be a linear transformation. If $v_1, v_2, v_3 \in V$ are so that

$$T(v_1) = 2, \quad T(v_2) = 3 \quad \text{and} \quad T(v_3) = 1,$$

then $v_1 - v_2 + v_3$ is in $\text{Ker}(T)$.

TRUE FALSE

Question 22: Let $u_1, \dots, u_k \in \mathbb{R}^n$ be orthonormal vectors, with $0 < k < n$.

If $W = \text{span}\{u_1, \dots, u_k\}$ and $v \neq 0$ is a vector in W^\perp , then $\text{span}\{v, u_1, \dots, u_k\}$ is a subspace of \mathbb{R}^n of dimension $k + 1$.

TRUE FALSE

Question 23: Let V be a vector space and let v_1, v_2, v_3 be three vectors in V . If the set $\{v_1, v_2, v_3\}$ is linearly independent, then the set $\{v_1 - v_2, v_1 - v_3, v_2 - v_3\}$ is linearly independent as well.

TRUE FALSE

Question 24: The number of non-zero rows in the reduced echelon form of a matrix is equal to the rank of the matrix.

TRUE FALSE

Question 25: Let A be a matrix of size $m \times n$ and let B be a matrix of size $n \times m$. If AB is invertible, then both A and B are also invertible.

TRUE FALSE

Question 26: Let A and B be two $n \times n$ matrices. If A and B are both symmetric, then $A + B$ is diagonalizable.

TRUE FALSE

Question 27: Let A be a square matrix of size 3×3 . If $p(\lambda) = -\lambda(1 - \lambda)(2 - \lambda)$ is the characteristic polynomial of A , then $\text{rank}(A) = 2$.

TRUE FALSE

Question 28: If $A, B \in \mathbb{R}^{n \times n}$ are so that $(A + B)^T = A + B$, then A and B are symmetric.

TRUE FALSE

Third part: open questions

- Answer in the empty space below using a black or dark blue ballpen.
- Your answer should be carefully justified, and all the steps of your argument should be discussed in details.
- Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 3 points.*

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Do not write here

Let A be a matrix of size $n \times n$ and let $u, v, w \in \mathbb{R}^n$ so that Au, Av, Aw are linearly independent.

Prove that u, v, w are also linearly independent.



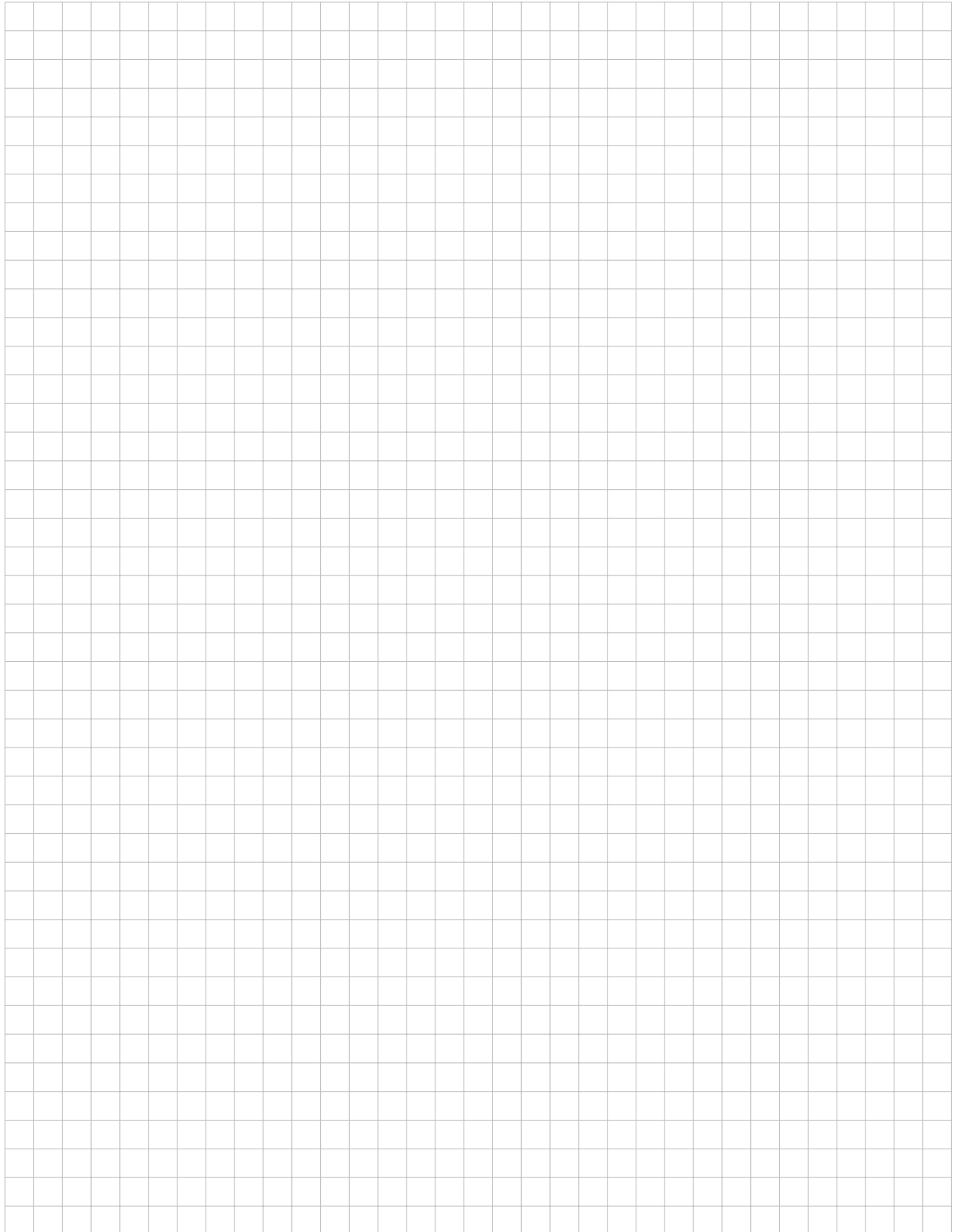
Question 30: *This question is worth 3 points.*

_0 _1 _2 _3

Do not write here

Let A be a matrix of size $n \times n$ satisfying $A^3 = I_n$.

- (a) Find $\text{Ker}(A)$.
- (b) Let $\lambda \in \mathbb{R}$ be an eigenvalue of A . Compute λ .



Question 31: *This question is worth 4 points.*

_0 _1 _2 _3 _4

Do not write here

Let $A = \begin{pmatrix} 2025 & 1 & 1 & 1 \\ 1 & 2025 & 1 & 1 \\ 1 & 1 & 2025 & 1 \\ 1 & 1 & 1 & 2025 \end{pmatrix} \in \mathbb{R}^{4 \times 4}$ and let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^4$.

- (a) Show that v is an eigenvector of A and find the corresponding eigenvalue.
- (b) Find a basis for the eigenspace of the eigenvalue $\lambda = 2024$.
- (c) Find a diagonal matrix D that is similar to A .

